GAUSS DECOMPOSITION FOR QUANTUM GROUPS AND DUALITY

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Abstract

The Gauss decomposition of quantum groups and supergroups are considered. The main attention is paid to the R-matrix formulation of the Gauss decomposition and its properties as well as its relation to the contraction procedure. Duality aspects of the Gauss decomposition are also touched. For clarity of exposition a few simple examples are considered in some details.

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1 Introduction

In this work using elementary algebraic methods and the R-matrix approach [1] we consider the Gauss decomposition of a q-matrix T (that is a matrix whose elements are the generators of the considered quantum group) $T = T_L T_D T_U$ in strictly lower- and upper-triangular matrices (with units on their diagonals) and a diagonal matrix T_D. Such a decomposition of a given matrix (with non commuting entries) into the product of matrices of the special type (similar to the Gauss decomposition) is the particular case of the general factorization problem [2], which can be considered as a cornerstone for many constructions of the classical as well as quantum inverse scattering methods. It should be pointed out that in different contexts such decompositions can be found (sometimes in non explicit form) in many papers on quantum deformations (see [5] for more details and references).

In quite a general framework of the quantum double construction the Gauss decomposition was considered in [3], where the universal triangular objects $\mathcal{M}^{\pm}$ were defined. Their matrix representations on one of the factors $(\rho \otimes 1)\mathcal{M}^{\pm} = M^{\pm}$ are the solutions of the FRT-relations

$$RM_1^{\pm} M_2^{\pm} = M_2^{\pm} M_1^{\pm} R, \quad M_1^{+} M_2^{-} = M_2^{-} M_1^{+}.$$

Their product $M^- M^+$ after the unification of diagonal elements ($M_{ii}^- M_{ii}^+ = A_{ii}$) gives the $q-$matrix T. Usually the new generators, defined by the Gauss decomposition, have simpler commutation rules (multiplication) but more complicated expressions for the coproduct.

The simpler structure of the commutation rules for the Gauss decomposition generators of the quantum groups simplifies [4] the problem of their q-bosonization (that is their realization by the creation and annihilation operators of the quantum deformed oscillator [6]-[7]). For the dual objects

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– quantum algebras (or quantum deformations of universal enveloping algebras of the classical Lie algebras) this problem was considered in [8, 9] and for q-superalgebras in [10, 11].

The paper is set up as follows. The general properties of the Gauss decomposition are treated in Sec.2 in the form most suitable for the best known case of the quantum group $GL_q(n)$, but general methods (after obvious modifications) are valid also for other quantum groups and even supergroups. It also concerns the remarks given in this section about, for example, the contraction procedure and duality considerations valid in other cases. Some peculiarities of the orthogonal and symplectic quantum groups and supergroups are discussed in Sec.3. The developed methods are briefly illustrated in Sec.4 where results of the Gauss decomposition for the quantum groups $SO_q(3)$, $Sp_q(2)$ and for some supergroups are given. We refer to the works [5],[12] for more details.

2 Gauss factorization for the quantum group $GL_q(n)$

According to the FRT-approach the matrix relation

$$RT_1T_2 = T_2T_1R, \tag{1}$$

encodes quadratic (commutation) relations between quantum group (QG) generators. The R-matrix for $GL_q(n)$ (A_l series) is the lower-triangular numerical $n^2 \times n^2$ -matrix [1]:

$$R = \sum_{i\neq j}^{n} E_{ii} \otimes E_{jj} + q\sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \lambda \sum_{i<j}^{n} E_{ji} \otimes E_{ij}, \tag{2}$$

where $\lambda = q - q^{-1}, q \neq 1$ and E_{kl} is the matrix unit, that is the $n \times n$-matrix with the only nonzero element $(E_{kl})_{kl} = 1$. With the R-matrix thus defined the FRT-equation gives the following commutation rules:

$$t_{ij}t_{ik} = qt_{ik}t_{ij}, \qquad t_{ik}t_{lj} = t_{lj}t_{ik}, \quad t_{ik}t_{lk} = qt_{lk}t_{ik}, \quad [t_{ij}, t_{lk}] = \lambda t_{ik}t_{lj}, \tag{3}$$

where $1 \leq j < k \leq n,\ 1 \leq i < l \leq n$. From these relations it follows that the algebra F_q has a rich subalgebra structure. In particular, any four matrix elements t_{ij}, t_{kj}, t_{il}, t_{kl} $\quad (1 \leq i, j, k, l \leq n)$ (in other words, any four elements standing at intersections of two arbitrary rows and two arbitrary columns) generate a $GL_q(2)$ -subalgebra.

The quantum determinant (q-det) is the expression [1]

$$D_q(T) = \det_q T = \sum_{\sigma} (-q)^{l(\sigma)} t_{1\sigma(1)} t_{2\sigma(2)} \cdot \ldots \cdot t_{n\sigma(n)}, \tag{4}$$

where $l(\sigma)$ is the length of a substitution $\sigma \in \mathbf{S}_n$. Note that $\det_q T$ is the central element for $GL_q(n)$ [1]. In the $GL_q(2)-$case q-det is equal to

$$\det_q T = t_{11}t_{22} - qt_{12}t_{21}.$$

Invertibility of $\det_q T$ is the necessary condition for endowing the algebra $F_q = \mathrm{Fun}(GL_q(n))$ with a Hopf algebra structure [1]. We shall suppose this condition to be fulfilled in what follows.

The Gauss decomposition for a q-matrix T is

$$T = T_L T_D T_U = T_L T^{(+)} = T^{(-)} T_U, \tag{5}$$

where $T_L = (l_{ik})$ is a strictly lower-triangular matrix (with the units at the main diagonal: $l_{kk} = 1$); $T_D = \mathrm{diag}(A_{kk})$; $T_U = (u_{ik})$ is a strictly upper-triangular matrix ($u_{kk} = 1$); $T^{(+)} := T_D T_U$ and $T^{(-)} = T_L T_D$.

The Gauss algorithm of triangularization of classical number matrices can be applied, as well, for quantum groups. Introduce a strictly lower-triangular $n \times n$-matrix W_L, which maps the q-matrix T into an upper-triangular matrix $T^{(+)}$:

$$W_L T = T^{(+)}. \tag{6}$$

As we wish the upper triangularity of the matrix $T^{(+)}$, the condition (6) gives the system

$$w_k T_{(k-1)} = -t_k \tag{7}$$

of linear equations for elements of a row $w_k = (w_{k,1}, w_{k,2}, ..., w_{k,k-1})$ with generators of the algebra F_q as coefficients. In this system $T_{(k)}$ denotes a matrix obtained from T by omitting the last $(n-k)$ rows and $(n-k)$ columns. If the q-matrix $T_{(k)}$ is invertible, then the solution of this system has the form

$$w_k = -t_k T^{-1}_{(k-1)}.$$

The elements of the inverse matrix are given by the formulae [1]

$$(T^{-1})_{ij} = (-q)^{i-j} (\det_q T)^{-1} M_q(i,j). \tag{8}$$

where $M_q(i,j)$ is the q-minor, that is the q-det of the matrix which is obtained from T by omitting of the ith row and jth column. Thus, all of the W_L-matrix elements are uniquely defined. The condition (6) allows us to find all the nonzero elements of the matrix $T^{(+)}$. In particular the diagonal elements (except $(T^{(+)})_{11} = t_{11}$) have the form

$$(T^{(+)})_{ii} = (\det_q T_{(i-1)})^{-1} (\det_q T_{(i)}). \tag{9}$$

Note that the last relation is a direct q-analog of the classical formula. It is easy to see that the diagonal elements $(T^{(+)})_{ii}$ of the matrix $T^{(+)}$ are mutually commuting. With the help of the commutation relations between the elements t_{ij} of the matrix T and their main minors we can find the commutation relations between the elements $(T^{(+)})_{ij}$ of the matrix $T^{(+)}$.

The matrix W_L is strictly-lower triangular, so it can be inverted. Elements of the inverse matrix $(T_L) = (W_L)^{-1}$ are polynomials on the elements of the matrix W_L. The relation (6) provides the desired decomposition

$$T = T_L T^{(+)}. \tag{10}$$

In close analogy with the previous case, one can (using an operator W_U) define the decomposition

$$T = T^{(-)} T_U. \tag{11}$$

Multiplying from the right the relation (10) by the matrix W_U, $T^{(-)} = TW_U$, we obtain

$$T^{(-)} = TW_U = T_L T^{(+)} W_U = T_L(T^{(+)} T_U^{-1}) = T_L T_D,$$

where $T_D = T^{(+)} T_U^{-1}$ is the diagonal matrix with elements of the form given by the formula (9). In the same manner we obtain $T^{(+)} = T_D T_U$. As a result we have the Gauss factorization (5) of the q-matrix $T = T_L T_D T_U$.

In principle the factorization procedure considered above allows us to get the expressions for elements of all the matrices participating in the decomposition in terms of the original generators of $GL_q(n)$. Moreover, the procedure defines the explicit form of $\det_q T$ and minors and commutation rules for them. The latter is especially important for the QGs of series other than A_n. Unfortunately, such a factorization procedure is rather cumbersome. This difficulty can be avoided with the help of the contraction procedure considered, for example, in [13, 14] for the case of QGs.

In the fundamental representation the Cartan elements h_i of $gl(n)$ are realized by the $n \times n$ matrices $h_i = \frac{1}{2n} E_{ii}$. They are not changed by standard quantum deformations. Moreover, they remain primitive, that is, their coproducts have the form $\Delta(h_i) \equiv H_i = h_i \otimes 1 + 1 \otimes h_i$. Due to the cocommutativity of Cartan elements ($\Delta(h_i) = \Delta'(h_i)$ with $\Delta' = P \circ \Delta$) we have

$$[R, H_i] = 0. \tag{12}$$

Let us subject the FRT-relation (1) to the similarity transformation with the matrix

$$K_\gamma = \exp(\sum_{i=1}^n \gamma_i h_i) \otimes \exp(\sum_{i=1}^n \gamma_i h_i) = \exp(\sum_{i=1}^n \gamma_i \Delta(h_i)) = \exp(\sum_{i=1}^n \gamma_i H_i), \tag{13}$$

where the numerical coefficients γ_i are strictly ordered $\gamma_1 > \gamma_2 > \ldots > \gamma_n > 0$. In view of the relation (12), such a transformation affects only the matrices T_l, $(l = 1, 2)$:

$$T \longrightarrow K_\gamma T K_\gamma^{-1} = \mathrm{diag}(e^{\gamma_i}) T \mathrm{diag}(e^{-\gamma_i}), \qquad t_{ij} \longmapsto t_{ij} e^{\gamma_i - \gamma_j}. \tag{14}$$

Introduce the two sets of new generators

$$t_{ij}^{(+)} = \begin{cases} t_{ij} e^{\gamma_i - \gamma_j} & i \le j \\ t_{ij} & i > j \end{cases} ; \quad t_{ij}^{(-)} = \begin{cases} t_{ij} & i < j \\ t_{ij} e^{\gamma_i - \gamma_j} & i \ge j \end{cases} .$$

Let $\gamma_i - \gamma_j = \gamma_{ij} \varepsilon$, where $\gamma_{ij} > 0$, if $i < j$ and $\gamma_{ij} < 0$, if $i > j$. When $\varepsilon \to \infty$ ($\varepsilon \to -\infty$) in the set $\{t_{ij}^{(+)}\}$ $(\{t_{ij}^{(-)}\})$ all the matrix elements with $i > j$ $(i < j)$ vanish. Thus, we have constructed the homomorphisms of the algebra F_q into the algebras $F_q^{(\pm)}$, generated by the elements of the upper- and lower-triangular q-matrices. In this case the commutation rules for the new generators are completely defined by the initial R-matrix:

$$R T_1^{(\pm)} T_2^{(\pm)} = T_2^{(\pm)} T_1^{(\pm)} R. \tag{15}$$

Note that for contracted algebras (and their generators) we use the same notations as in the case of algebras obtained by triangularization procedure. It can be verified that commutation relations for the Gauss generators are uniquely defined by the R-matrix.

Similar contraction procedure allows us to find the homomorphisms $T^{(\pm)} \rightarrow A$ of the QGs described by $T^{(\pm)}$ into the group whose generators are the elements of the diagonal matrix T_D. The multiplication rules for the latter group are also determined by the FRT-relation $RA_1A_2 = A_2A_1R, \qquad (A_{ij} = \delta_{ij}A_{ij})$, which, in view of the structure of the R-matrix for the quantum group $GL_q(n)$, is equivalent to the relation

$$A_1A_2 = A_2A_1. \tag{16}$$

The relation (16) means the commutativity of the elements of the diagonal quantum group.

In the limit $\epsilon \rightarrow \infty$ ($\epsilon \rightarrow -\infty$) the transformation (13) can be treated not only as a factorization of F_q but also as a contraction procedure for the Hopf algebras: $F_q \rightarrow F_q^{\mathrm{contr}}$. In terms of generators $\left\{t_{ij}^{(+)}\right\}$ (respectively $\left\{t_{ij}^{(-)}\right\}$) this contraction is described by the following transformation:

$$t_{ij}^{(+)} \rightarrow \begin{cases} t_{ij}^{(+)}; & i \leq j \\ e^{\gamma_{ij}} t_{ij}^{(+)}; & i > j \end{cases} \quad \text{or} \quad t_{ij}^{(-)} \rightarrow \begin{cases} e^{\gamma_{ij}} t_{ij}^{(-)}; & i < j \\ t_{ij}^{(-)}; & i \geq j \end{cases} \tag{17}$$

We have seen that for the set of generators $\left\{t_{ij}^{(+)}\right\}$ on the factorspace the multiplication is still defined by the RTT-equation with the same R . The limiting transition (17) allows one to examine the whole set of generators and it is easily seen that <u>all</u> the structure constants of F_q described by the relations (3) have the finite limits when ϵ tends to $+\infty$ (respectively $-\infty$). Obviously the co-structure constants of F_q also have the finite limit values. Consider, for example, the co-algebra F_q^{contr} for the first type of transformations (17):

$$\Delta t_{ij}^{(+)} = \begin{cases} \sum\limits_{i \leq s \leq j} t_{is}^{(+)} \otimes t_{sj}^{(+)} & \text{for} \quad i < j \\ \sum\limits_{j \leq s \leq i} t_{is}^{(+)} \otimes t_{sj}^{(+)} & \text{for} \quad j < i \end{cases} \tag{18}$$

The coproduct here does not mix the upper and lower parts of the T-matrix. Note that while in (15) the limiting procedure nullifies one of the triangular parts of the initial T-matrix, the relations (3) rewritten for $t_{ij}^{(+)}$ still describe the multiplication rules for all the generators of F_q^{contr}. The contraction does not touch the compositions of the first three types in (3). To fix the new relations of the fourth type let us introduce the grading function σ.

$$\sigma\left(t_{ij}^{(+)}\right) = \begin{cases} +1 & \text{for} \quad i < j \\ 0 & \text{for} \quad i = j \\ -1 & \text{for} \quad i > j \end{cases} .$$

For $1 \leq j < k \leq n$ and $1 \leq i < l \leq n$ the last commutator in (3) has the following values:

$$\left[t_{ij}^{(+)}, t_{lk}^{(+)}\right] = \begin{cases} 0 & \begin{cases} \text{for} & \sigma\left(t_{ij}^{(+)}\right) \neq \sigma\left(t_{lk}^{(+)}\right) \\ \text{for} & \sigma\left(t_{ij}^{(+)}\right) = \sigma\left(t_{lk}^{(+)}\right), \sigma\left(t_{ik}^{(+)}\right) \neq \sigma\left(t_{lj}^{(+)}\right) \end{cases} \\ \lambda t_{ik}^{(+)} t_{lj}^{(+)} & \quad \text{for } \sigma\left(t_{ij}^{(+)}\right) = \sigma\left(t_{lk}^{(+)}\right), \sigma\left(t_{ik}^{(+)}\right) = \sigma\left(t_{lj}^{(+)}\right) \end{cases} \tag{19}$$

The Hopf algebra F_q^{contr} describes a QG. The corresponding quantum algebra can be defined on the space dual to that of F_q^{contr}. To obtain the appropriate form of relations for the dual Hopf

decomposition (5) one can see that the transformations (13) and (14) have in this basis the same form:

$$\begin{cases} l_{ik} \longrightarrow l_{ik} e^{\gamma_{ik}} \equiv l_{ik}^{(+)} \\ u_{ik} \longrightarrow u_{ik} e^{\gamma_{ik}} \equiv u_{ik}^{(+)} e^{\gamma_{ik}} \\ A_{kk} \longrightarrow A_{kk} \equiv A_{kk}^{(+)} \end{cases} \tag{20}$$

Let us introduce on the space dual to that of F_q the generators $\{\mu_{i,i+1}, \nu_{j+1,j} \alpha_{kk}\}$,

$$\begin{array}{ccc} \left\langle \alpha_{kk}, A_{ss}^{(+)} \right\rangle = \delta_{ks}; & \left\langle \alpha_{kk}, u_{i,i+1}^{(+)} \right\rangle = 0; & \left\langle \alpha_{kk}, l_{j+1,j}^{(+)} \right\rangle = 0; \\ \left\langle \mu_{i,i+1}, A_{ss}^{(+)} \right\rangle = 0; & \left\langle \mu_{k,k+1}, u_{s,s+1}^{(+)} \right\rangle = \delta_{ks}; & \left\langle \mu_{i,i+1}, l_{j+1,j}^{(+)} \right\rangle = 0; \\ \left\langle \nu_{i,i+1}, A_{ss}^{(+)} \right\rangle = 0; & \left\langle \nu_{i,i+1}, u_{i,i+1}^{(+)} \right\rangle = 0; & \left\langle \nu_{i,i+1}, l_{j+1,j}^{(+)} \right\rangle = \delta_{ij}. \end{array} \tag{21}$$

These elements form the Chevalley basis for $U_q(gl(n))$ with the simple roots $\{\lambda_i\}$ and the defining relations [15]

$$[\alpha_{ii}, \alpha_{kk}] = 0; \quad [\alpha_{ii}, \mu_{j,j+1}] = (\lambda_i, \lambda_j)\, \mu_{j,j+1}; \quad [\alpha_{ii}, \nu_{j+1,j}] = -(\lambda_i, \lambda_j)\, \nu_{j+1,j}; \tag{22}$$

$$[\mu_{j,j+1}, \nu_{k+1,k}]_{e^{h/2\langle \lambda_j, \lambda_k \rangle}} = \delta_{jk} \frac{e^{-h\alpha_{jj}} - 1}{e^{-h} - 1}; \tag{23}$$

$$(\mathrm{ad}\mu_{j,j+1})^{1-\langle \lambda_j, \lambda_k \rangle} \mu_{k,k+1} = 0; \qquad (\mathrm{ad}\nu_{j,j+1})^{1-\langle \lambda_j, \lambda_k \rangle} \nu_{k,k+1} = 0; \tag{24}$$

$$\begin{gathered} \Delta\alpha_{ii} = \alpha_{ii} \otimes 1 + 1 \otimes \alpha_{ii}; \\ \Delta\mu_{j,j+1} = \mu_{j,j+1} \otimes 1 + e^{-\frac{h}{2}\alpha_{jj}} \otimes \mu_{j,j+1}; \qquad \Delta\nu_{j,j+1} = \nu_{j,j+1} \otimes 1 + e^{-\frac{h}{2}\alpha_{jj}} \otimes \nu_{j,j+1}; \end{gathered} \tag{25}$$

The transformation

$$\nu_{i,j} \longrightarrow e^{-\gamma_{ij}} \nu_{i,j}, \qquad \mu_{i,j} \longrightarrow \mu_{i,j}, \qquad \alpha_{ii} \longrightarrow \alpha_{ii} \tag{26}$$

is dual to (20) and describes a contraction $U_q(gl(n)) \longrightarrow U_q^{\mathrm{contr}}$ when $\epsilon \longrightarrow \infty$. In this limiting procedure the duality is preserved. The obtained Hopf algebra U_q^{contr} is dual to F_q^{contr}. The contraction changes only one of the defining relations: instead of (23) one obtains

$$[\mu_{j,j+1}, \nu_{k+1,k}]_{e^{h/2\langle \lambda_j, \lambda_k \rangle}} = 0; \tag{27}$$

Thus U_q^{contr} is the quantization of a Lie algebra $gl(n)^{\mathrm{contr}}$. The latter has the same Borel subalgebras $b^+$and b^- as the original $gl(n)$ but here the corresponding n^+ and n^- subalgebras commute:

$$\left[n^+, n^-\right] = 0 \tag{28}$$

This commutativity is in total accordance with the "separation" of coproducts in (18).

We have thus demonstrated that the contraction described by the relations (13),(14) leads to the QG F_q^{contr} which is the quantized algebra of functions over the Lie group $GL^{\mathrm{contr}}(n,\mathrm{C})$ with the Lie algebra $gl^{\mathrm{contr}}(n,\mathrm{C})$ defined by the relations (22), (24-25) and (27).

The contraction of the type (26) can be performed for an arbitrary simple Lie algebra. It is sufficient to multiply every element x_λ of b^+ (or b^-) by $\epsilon^{-\sum m_i}$ (respectively $\epsilon^{\sum m_i}$) and go to the limit $\epsilon \longrightarrow \infty$. Here m_i are the coordinates of the root λ in terms of simple roots $\{\lambda_i\}$. Such a contraction always exists and forces subalgebras $n^+$and n^- to commute.

Considering contractions performed by the operators $K_{1\gamma} = \exp(\sum_{i=1}^{n} \gamma_i h_i) \otimes 1$ and $K_{2\gamma} = 1 \otimes \exp(\sum_{i=1}^{n} \gamma_i h_i)$, we obtain the following relations (where R_D is the diagonal part of the R-matrix).

$$R_D A_1 T_2^{(-)} = T_2^{(-)} A_1 R_D; \quad R_D T_1^{(+)} A_2 = A_2 T_1^{(+)} R_D; \quad R_D T_1^{(+)} T_2^{(-)} = T_2^{(-)} T_1^{(+)} R_D. \tag{29}$$

Other commutation rules for the generators of the QGs T_U and T_L can be obtained from the relations (5,15-29):

$$\begin{array}{cc} R R_D T_{U1} R_D^{-1} T_{U2} = R_D T_{U2} R_D^{-1} T_{U1} R, & R T_{L1} R_D T_{L2} R_D^{-1} = T_{L2} R_D T_{L1} R_D^{-1} R, \\ R_D T_{D1} T_{L2} = T_{L2} T_{D1} R_D, & R_D T_{U1} T_{D2} = T_{D2} T_{U1} R_D, \end{array} \tag{30}$$

(and similar equalities with interchange $1 \leftrightarrow 2$). If we separate the T_U and T_L parts from $T^{(\pm)}$ in (29) and take into account relations (30) we get

$$T_{U1} T_{L2} = T_{L2} T_{U1}. \tag{31}$$

Thus, all the elements of the q-matrix T_U commute with every element of the q-matrix T_L. Because of the relation $[R, R_D] = 0$, which is valid for $GL_q(n)$, $Sp_q(n)$ and $SO_q(2n)$ (but not for $SO_q(2n+1)$!), we can rewrite the first two equations in (30) in the form of the reflection equation

$$R T_{U1} R_D^{-1} T_{U2} = T_{U2} R_D^{-1} T_{U1} R, \quad R T_{L1} R_D T_{L2} = T_{L2} R_D T_{L1} R.$$

The relations (5,15,29,30,31) together with easily deduced ones

$$R_D T_1^{(+)} T_{L2} = T_{L2} R_D T_1^{(+)}; \qquad T_2^{(-)} {R_D}^{-1} T_{L1} = T_{L1} T_2^{(-)} {R_D}^{-1}, \tag{32}$$

supply us with the full sets of commutation rules imposed on the elements of the matrices $\{T_L, T^{(+)}\}$, $\{T^{(-)}, T_U\}$ and $\{T_L, T_D, T_U\}$. This allows to consider each of these sets of elements as the new basis of generators for $GL_q(n)$. The basis $\{T_L, T_D, T_U\}$, which has the maximal number (for $N \geq 3$) of mutually commuting elements is a particularly convenient. Diagonal elements of T_D play the role of Cartan generators of a Lie algebra. Q-det (4) is the central element of $GL_q(n)$ and, as in the case of numerical matrices, can be written as the product of the diagonal matrix elements

$$\det_q T = \prod_{i=1}^{n} (T_D)_{ii}. \tag{33}$$

Of course, $\det_q T$ commutes with all the new generators just as well as with the original ones.

In conclusion of the Sec. we note that further decomposition of T_L and T_U is possible. Such a procedure leads to a new basis in $GL_q(n)$ with Weyl-type commutation relations for its elements [23].

3 Orthogonal and symplectic quantum groups, quantum supergoups

Commutation relations for generators of the orthogonal and symplectic QGs are determined not only by FRT-equation with the relevant R-matrix, but also by supplementary conditions [1]

$$TCT^tC^{-1} = 1, \qquad CT^tC^{-1}T = 1, \tag{34}$$

which are quantum analog of the known conditions for matrices of the orthogonal and symplectic Lie groups in the defining representation. In (34) T^t is the transposition of T, C is a numerical matrix [1] . Among the supplementary conditions encoded by (34) only those that have the nonzero right hand side are linearly independent with respect to the commutation relations defined by the FRT-equation (1).

Similarly to the case of $GL_q(n)$ in the series B_n, C_n and D_n (with the only exception of B_1) the corresponding quantized algebras of coordinate functions F_q have rich subalgebra structures. For example, a $GL_q(k)$-subalgebra is generated by the elements of the q-matrix T located at the intersections of rows with indices $(i_1, i_2, .., i_k)$ and columns with indices $(j_1, j_2, .., j_k)$ (when $k \leq n$ and no pairs of the type (i_m, i_l) with $i_m = (i_l)' = N + 1 - i_l$ occur in the set $\{i_p\}$ and also in $\{j_p\}$). The products of the generators belonging to the different $GL_q(k)$-subalgebras have much more complicated form.

The Gauss factorization of orthogonal, symplectic groups and supergroups can be carried out by the method described in section 2 for $GL_q(n)$. The corresponding commutation relations are governed by the same formulae ((5,15,29,30,31,(32). One must take into consideration that the supplementary conditions yield some new relations. For instance, T_D-matrix generators satisfy the relations

$$(T_D)_{ii}(T_D)_{i'i'} = 1, \quad 1 \leq i \leq N. \tag{35}$$

Using the multiplication rules one can prove that the supplementary conditions for the matrices $T^{(\pm)}$ have the same form (34) as for T. It should be noted, that with respect to the elements of T_L (or T_U) these conditions contain not only quadratic but also linear terms. It allows us to exclude dependent generators from the generator list. The number of remaining generators is equal to the dimension of the corresponding Lie group. We shall illustrate such a reduction in the next section.

4 Examples

In this Sec. some details of the Gauss factorization of the symplectic and orthogonal QGs are considered using $\mathrm{SO}_q(3)$ and $Sp_q(2)$ as examples. The last part of this Sec. is devoted to the exposition of some simplest examples of quantum supergroups. The R-matrices for these groups have more nonzero matrix elements in comparison with the R-matrix for $GL_q(n)$ of the same rank. The complicated structure of the R-matrix causes additional complexification of the commutation relations (even for the already mentioned simplest cases the corresponding list of relations is rather long and cannot be presented here). Gauss decomposition changes this situation drastically. It provides the reduced list of generators, containing only independent ones, with rather simple commutation rules.

4.1 Quantum group $B_1 \sim SO_q(3)$

Let us apply the Gauss factorization procedure to the q-matrix of $SO_q(3)$

$$T = T_L T_D T_U = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} \tag{36}$$

After cumbersome computations one finds that here all the matrix elements can be described in terms of three independent generators u, A and z (just as in the classical Lie case):

$$\begin{gathered} A_{11} = A, \qquad A_{22} = 1, \qquad A_{33} = A^{-1}; \\ u_{12} = t_{11}^{-1} t_{12} = u, \quad u_{13} = t_{11}^{-1} t_{13} = -\mu^{-1} u^2, \quad u_{23} = -q^{1/2} u; \\ l_{21} = t_{21} t_{11}^{-1} = z, \quad l_{31} = t_{31} t_{11}^{-1} = -\mu^{-1} z^2, \quad l_{32} = -q^{-1/2} z. \end{gathered} \tag{37}$$

This gives

$$T = T_L T_D T_U = \begin{pmatrix} A & Au & -\mu^{-1} A u^2 \\ zA & I + zAu & -\mu^{-1} z A u^2 - q^{1/2} u \\ -\mu^{-1} z^2 A & -\mu^{-1} z^2 A u - q^{-1/2} z & \mu^{-2} z^2 A u^2 + zu + A^{-1} \end{pmatrix} \tag{38}$$

It is easy to see, that these new generators are subject to the following Weyl-like commutation rules

$$Au = quA, \quad Az = qzA, \quad uz = zu. \tag{39}$$

As a simple application of these results let us show that the q-det for the QG is equal to unity. We propose the following notations for minors of the quantum matrix T

$$\Delta \left[{}^{i\,j}_{k\,l}; q^{\alpha} \right] \equiv t_{ik} t_{jl} - q^{\alpha} t_{il} t_{jk}, \tag{40}$$

Then the q-det can be written in a simple form

$$\det{}_q T = t_{11} \Delta \left[{}^{2\,3}_{2\,3}; q \right] - t_{12} \Delta \left[{}^{2\,3}_{1\,3}; q^2 \right] + q t_{13} \Delta \left[{}^{2\,3}_{1\,2}; q \right] \tag{41}$$

In [16] the same expression was justified by geometric considerations. If we rewrite it in terms of new generators we easily get

$$\det{}_q T = 1. \tag{42}$$

This result agrees with the general statement made for QGs of classical types(see Subsec. 2.3 above): the q-det is equal to the product of diagonal elements of the matrix T_D.

The single matrix element can also be written as $t_{ij} = \Delta \left[{}^{i}_{j} \right]$. In these terms the nontrivial elements of T_L and T_U are the ratios of q-minors in complete analogy with classical (commutative) situation. For example,

$$T_U = \begin{pmatrix} 1 & \Delta^{-1} \left[{}^1_1 \right] \Delta \left[{}^1_2 \right] & \Delta^{-1} \left[{}^1_1 \right] \Delta \left[{}^1_3 \right] \\ 0 & 1 & \Delta^{-1} \left[{}^{2\,1}_{1\,2}; q \right] \Delta \left[{}^{2\,1}_{1\,3}; q^2 \right] \\ 0 & 0 & 1 \end{pmatrix} \tag{43}$$

4.2 Quantum group $C_2 \sim Sp_q(2)$

To realize the Gauss algorithm we must find the operators W_L and W_U (see Sec.2). Solving, for example, the system of equations (6) for W_L one has (in notation introduced in (40))

$$w^L_{21} = -t_{21} \Delta_q^{-1} [{}^1_1]; \quad w^L_{31} = (\Delta_q [{}^{2\,3}_{1\,3}; q] - \lambda \Delta_q [{}^{1\,4}_{1\,2}; q]) \Delta_q^{-1} [{}^{1\,2}_{1\,2}]; \quad w^L_{32} = -\Delta_q [{}^{1\,3}_{1\,2}; q] \Delta_q^{-1} [{}^{1\,2}_{1\,2}; q]; \tag{44}$$

Thus, we obtain

$$T^{(+)} = \begin{pmatrix} \Delta_q[{}^1_1] & t_{12} & t_{13} & t_{14} \\ 0 & \Delta_q^{-1}[{}^1_1]\Delta_q[{}^{1\,2}_{1\,2};q] & \Delta_q^{-1}[{}^1_1]\Delta_q[{}^{1\,2}_{1\,3};q] & \Delta_q^{-1}[{}^1_1](t_{11}t_{24}-q^2t_{14}t_{21}) \\ 0 & 0 & \Delta_q^{-1}[{}^{1\,2}_{1\,2};q]\Delta_q[{}^1_1] & -\Delta_q^{-1}[{}^1_1]t_{12} \\ 0 & 0 & 0 & \Delta_q^{-1}[{}^1_1] \end{pmatrix}$$

Here in close analogy with the commutative case, the diagonal elements are the ratios $\Delta_q^{-1}[{}^{1,2,\ldots,k-1}_{1,2,\ldots,k-1}]\Delta_q[{}^{1,2,\ldots,k}_{1,2,\ldots,k}]$ of the diagonal minors. In the matrix given above these ratios are further simplified using particular properties of the symplectic groups. Thus evaluating the matrix elements of $T^{(+)}$ on the places related to the diagonal GL_q-minors one fins the expressions of the form

$$D_q^{sp}[{}^{1,2,\ldots,k}_{1,2,\ldots,k}] = \sum_\sigma (-q)^{l(\sigma)} q^{l'(\sigma)} t_{1,\sigma(1)} t_{2,\sigma(2)} \cdot \ldots \cdot t_{k,\sigma(k)}, \tag{45}$$

with additional factor $q^{l'(\sigma)}$ in comparison with (4). Here $l'(\sigma)$ is the number of transpositions of "specific" elements (the transposition index). For example, $l'(1,2,4,3) = 0$, but $l'(1,3,2,4) = 1$ (2 and $3 = 2'$ are transposed). For $i = 3,4$ the following simplifications become possible in (45),

$$D_q^{sp}[{}^{1,2,3}_{1,2,3}] = D_q^{sp}[{}^1_1] \qquad D_q^{sp}(T) = D_q^{sp}[{}^{1,2,3,4}_{1,2,3,4}] = 1. \tag{46}$$

So, in the symplectic case it is natural to define the q-det by the formula (45). This definition is in agreement with the one presented in [16] with the different argumentation.

The non zero elements of the matrix $W_L^{-1} = T_L$ have the form (remind that $T_L = (l_{ij}), T_D = (A_{ii}), T_U = (u_{ij})$)

$$\begin{gathered} (T_L)_{21} = -w_{21}; \quad (T_L)_{32} = -w_{32}; \quad (T_L)_{43} = -w_{43}; \\ (T_L)_{31} = w_{32}w_{21} - w_{31} = t_{31}D_q^{-1}[{}^1_1]; \\ (T_L)_{41} = -w_{43}w_{32}w_{21} + w_{43}w_{31} + w_{42}w_{21} - w_{41} = t_{41}D_q^{-1}[{}^1_1]; \\ (T_L)_{21} = w_{43}w_{32} - w_{42} = q^{-1}D_q[{}^{1,4}_{1,2}]D_q^{-1}[{}^{1,2}_{1,2}]; \end{gathered} \tag{47}$$

Using definition $T^{(+)} = T_D T_U$ matrix elements of $T^{(+)}$ can be obtained. Analogous procedure based on the matrix W_U leads, naturally, to the same results.

The obtained formulas allow us to find commutation relations for the new basic generators induced by the Gauss decomposition. The final form of these relations is

$$\begin{cases} [A_{kk}, A_{jj}] = 0 \\ [u_{kl}, l_{ij}] = 0 \end{cases} \qquad \begin{cases} A_m l_{ij} &=& q^{(\delta_{mj}-\delta_{mj'}-\delta_{mi}+\delta_{mi'})} l_{ij} A_m; \\ A_m u_{ij} &=& q^{(\delta_{im}-\delta_{im'}-\delta_{jm}+\delta_{jmi'})} u_{ij} A_m; \end{cases}$$

$$\begin{cases} l_{21}l_{31} &=& q^2 l_{31}l_{21} + q\lambda l_{41}; \\ l_{32}l_{21} &=& q^2 l_{21}l_{32} - (q^4-1)l_{31}; \\ l_{31}l_{32} &=& q^2 l_{32}l_{31}; \\ [l_{41}, l_{ij}] &=& 0; \end{cases} \qquad \begin{cases} u_{12}u_{13} &=& q^2 u_{13}u_{12} + q\lambda u_{14}; \\ u_{23}u_{12} &=& q^2 u_{12}u_{23} - (q^2-q^{-2})u_{13}; \\ u_{13}u_{23} &=& q^2 u_{23}u_{13}; \\ [u_{14}, u_{kl}] &=& 0; \end{cases}$$

The supplementary conditions cause the following constrains on the set of new generators

$$\begin{aligned} A_{33} &= A_{22}^{-1}; & A_{44} &= A_{11}^{-1}; \\ l_{42} &= q^2 l_{31} - l_{21} l_{32}; & l_{43} &= -l_{21}; \\ u_{24} &= q^2 (u_{13} - u_{12} u_{23}); & u_{34} &= -u_{12}; \end{aligned} \tag{49}$$

Hence, the number of independent generators decreases to ten, i.e. to the dimension of the corresponding classical Lie group.

4.3 Quantum supergroups

The FRT-relation for quantum supergroups has the same form as in (1), but the matrix tensor product includes additional sign factors ($\pm$) related to $\mathbf{Z}_2$-grading [17]. Z_2-graded vector space (superspace) decomposes into the direct sum of subspaces $V_0 \oplus V_1$ of even and odd vectors. The parity function ($p(v) = 0$ at $v \in V_0$ and $p(w) = 1$ at $w \in V_1$) is defined on them. As a rule, a vector basis with definite parity $p(v_i) = p(i) = 0, 1$ is used. In this basis the row and column parities are introduced in the matrix space $\mathrm{End}(V)$. The tensor product of two even matrices F, G ($p(F_{ij}) = p(i) + p(j)$) has the following signs [17]

$$(F \otimes G)_{ij;kl} = (-1)^{p(j)(p(i)+p(k))} F_{ik} G_{jl}. \tag{50}$$

Due to this prescription $T_2 = I \otimes T$ has the same block-diagonal form as in the usual (non super) case while $T_1 = T \otimes I$ includes the additional sign factor (-1) for odd elements standing at odd rows of blocks. For the $GL_q(n|m)$ quantum supergroup the R-matrix structure is the same as for the $GL_q(n+m)$ but at odd-odd rows q is changed by q^{-1}

$$R = \sum_{i,j} \left(1 - \delta_{ij}(1 - q^{1-2p(i)})\right) e_{ii} \otimes e_{jj} + \lambda \sum_{i>j} e_{ij} \otimes e_{ji}. \tag{51}$$

Let us remind that the tensor product notation in (51) refers to the graded matrices.

The same contraction procedure as in Sec.2 results in homomorphisms of $T = (t_{ij})$ onto $T^{(\pm)}$, T_D matrices and leads to the corresponding R-matrix relations. Let us give some of them emphasizing the peculiarities of the supergroup case.

Using the R-matrix block structure in the relations

$$R T^{(\pm)}{}_1 T^{(\pm)}{}_2 = T^{(\pm)}{}_2 T^{(\pm)}{}_1 R, \tag{52}$$

we can find commutation rules for the diagonal elements,

$$R_D (T_D)_1 T^{(-)}{}_2 = T^{(-)}{}_2 (T_D)_1 R_D, \quad R_D T^{(+)}{}_2 (T_D)_1 = (T_D)_1 T^{(+)}{}_2 R_D. \tag{53}$$

As above for the mutually commutative elements $A_{ii}:\ T_D = \mathrm{diag}(A_{11}, A_{22}, \ldots)$ one has

$$A_{ii} T^{(\pm)} A_{ii}^{-1} = (R_D)^{\pm 1}{}_{ii} T^{(\pm)} (R_D)^{\mp 1}{}_{ii}. \tag{54}$$

However, the diagonal block structure of R_D is different here. As a consequence the $GL_q(n|m)$ central element is the ratio of the two products corresponding to even and odd rows

$$\mathrm{s-det}_q T = \left(\frac{\prod_{i=1}^{n} A_{ii}}{\prod_{i=n+1}^{m} A_{ii}} \right). \tag{55}$$

It is naturally to call this expression the quantum superdeterminant (q -Berezinian).

For the supergroup $GL_q(1|1)$ the commutation relations of the q-matrix elements $T = \left(\begin{smallmatrix} a & \beta \\ \gamma & d \end{smallmatrix}\right)$ have the form [18]

$$\begin{array}{lll} a\beta = q\beta a, & \beta d = q^{-1}d\beta, & \beta\gamma = -\gamma\beta, \\ a\gamma = q\gamma a, & \gamma d = q^{-1}d\gamma, & ad = da + \lambda\gamma\beta, \end{array} \quad \beta^2 = \gamma^2 = 0. \tag{56}$$

Here we used the Greek letters for odd (nilpotent) generators. The Gauss decomposition gives

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varsigma & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}, \tag{57}$$

$$A = a, \quad \psi = A^{-1}\beta, \quad \varsigma = \gamma A^{-1}, \quad B = d - \gamma A^{-1}\beta. \tag{58}$$

The relations

$$\begin{array}{c} [A, B] = 0, \quad A\psi = q\psi A, \quad A\varsigma = q\varsigma A, \quad \psi^2 = \varsigma^2 = 0, \\ \psi\varsigma + \varsigma\psi = 0, \quad B\psi = q\psi B, \quad B\varsigma = q\varsigma B \end{array} \tag{59}$$

cause the centrality of the superdeterminant in $GL_q(1|1)$ [18]

$$\mathrm{s} - \det_q T = AB^{-1} = a^2(ad - q\gamma\beta)^{-1} = a/(d - \gamma a^{-1}\beta). \tag{60}$$

In the $GL_q(2|1)$ case the q-matrix of generators has the form

$$T = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} = \begin{pmatrix} M & \Psi \\ \Phi & f \end{pmatrix}. \tag{61}$$

The even M-matrix elements form $GL_q(2)$ subgroup with the commutation rules (4). The elements of each (2×2) submatrix with even generators at its diagonal form $GL_q(1|1)$ supersubgroup with (56) - type commutation relations. The remaining multiplications look like follows

$$\begin{array}{lllll} \alpha\beta = -q^{-1}\beta\alpha, & c\alpha = \alpha c, & b\gamma = \gamma b, & a\beta = \beta a + \lambda c\alpha, & b\beta = \beta b + \lambda d\alpha, \\ \gamma\delta = -q^{-1}\delta\gamma, & d\alpha = \alpha d, & d\gamma = \gamma d, & a\delta = \delta a + \lambda\gamma b, & c\delta = \delta c + \lambda\gamma d. \end{array}$$

The new generators produced by the Gauss decomposition

$$T = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ v & w & 1 \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \tag{62}$$

have the following commutation rules

$$\begin{array}{lll} Ax = qxA, & Ay = qyA, & Az = zA, \\ Au = quA, & Av = qvA, & Aw = wA, \end{array}$$

$$\begin{array}{lll} Bx = q^{-1}xB, & By = yB, & Bz = qzB, \\ Bu = q^{-1}uB, & Bv = vB, & Bw = qwB, \end{array}$$

$$\begin{array}{lll} Cx = xC, & Cy = qyC, & Cz = qzC, \\ Cu = uC, & Cv = qvC, & Cw = qwC \end{array}$$

$$[A,B]=[A,C]=[B,C]=0, \qquad y^2=z^2=v^2=w^2=0$$

$$xy=qyx, \qquad yz=-q^{-1}zy, \qquad qxz-zx=\lambda y,$$
$$uv=qvu, \qquad vw=-q^{-1}wv, \qquad uw-qw^{-1}=\lambda v,$$

$$[x,u]=[x,v]=[x,w]=0, \quad [u,x]=[u,x]=[u,y]=[u,z]=0,$$
$$yv+vy=0=yw+wy, \qquad zv+vz=0=zw+wz.$$

The superdeterminant

$$\mathrm{s}{-}det_qT=ABC^{-1}=det_qM/C$$

is a central element. The latter expression follows from the block Gauss decomposition of (61). In particular for the $GL_q(m|n)$ matrix T in the block form one has (cf. [19])

$$s-det_qT=det_qA/det_q(D-CA^{-1}B)$$

which is formally the standard expression.

Generalization of the above results to $GL_q(m|n)$ and other quantum supergroups looks rather straightforward. Although, as usual, quantum supergroup $OSp_q(1|2)$ with the rank one has its own peculiarities. In this case the q-matrix T has three independent generators while the undeformed supergroup has five. The T-matrix for this quantum supergroup in the fundamental co-representation has the dimension three and three diagonals of the 9x9 R-matrix diagonal blocks are $(q,1,1/q)$ $(1,1,1)$ and $(1/q,1,q)$. Hence, the diagonal elements $T_D=\mathrm{diag}(A,\ B,\ C)$ of its Gauss decomposition give rise to the central elements $AC=CA$ and B. The quantum super-determinant is $s-det_qT=AC/B=B$ due to the supplementary quadratic relation $T^{st}CT=\gamma C$ with $\gamma=B^2=AC=CA$ as in the case of the orthogonal and symplectic series. The lower-triangular matrix T_L (as well as upper-triangular T_U) has only one independent generator:

$$T_U=\begin{pmatrix}1 & x & x^2/\omega\\ 0 & 1 & -x/q^{1/2}\\ 0 & 0 & 1\end{pmatrix}, \qquad T_L=\begin{pmatrix}1 & 0 & 0\\ u & 1 & 0\\ u^2/\omega & q^{1/2}u & 1\end{pmatrix},$$

here $\omega=q^{1/2}-q^{-1/2}$. This fact is also reflected in the structure of the universal T-matrix : $\mathcal{T}=E_q^{(s)}(V_-u)exp(2Ha)E_{1/q}^{(s)}(V_+x)$, where x, u and $exp(a)=A$ are generators for the q-super-group $OSp_q(1|2)$, while V_- V_+ and H are the generators of the dual Hopf super-algebra $osp_q(1|2)$ and $E_q^{(s)}(t)$ is the q-exponent (see also [22], [23])

5 Conclusion

In this work we considered the Gauss decomposition of the quantum groups related to the classical Lie groups and supergroups by the elementary linear algebra and R-matrix methods. The Gauss factorization yields a new basis for these groups which is sometimes more convenient than the original one. Most of the relations for the Gauss generators are written in the $R-$matrix form. These commutation relations are simpler then the original rules. This is especially evident in the symplectic

definitions of B, C, D-series of quantum groups allow to extract the independent ones. Their number is equal to the dimension of the corresponding classical group. The Gauss factorization leads naturally to appearance of q-analogs of such classical notions as determinants, superdeterminants and minors. We also want to stress that the new basis is helpful for studies in quantum group representation theory. In particular as demonstrated in [4] it simplifies the q-bosonization problem. As was pointed out in the Introduction it looks that almost any relation and/or statement for standard matrices being appropriately "$q-$deformed" is valid for $q-$matrices.

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